RESEARCH ARTICLE OPEN ACCESS

Influence of MHD on Unsteady Helical Flows of Generalized Oldoyd-B Fluid between Two Infinite Coaxial Circular Cylinders

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ABSTRACT
Considering a fractional derivative model for unsteady magetohydrodynamic (MHD)helical flows of an Oldoyd-B fluid in concentric cylinders and circular cylinder are studied by using finite Hankel and Laplace transforms .The solution of velocity fields and the shear stresses of unsteady magetohydrodynamic (MHD)helical flows of an Oldoyd-B fluid in an annular pipe are obtained under series form in terms of Mittag –leffler function,satisfy all imposed initial and boundary condition , Finally the influence of model parameters on the velocity and shear stress are analyzed by graphical illustrations.

I. Introduction

Recently , a considerable attention has been devoted to the problem of how to predict the behavior of non-Newtonian fluid .The main reason for this is probably that fluids(such a molten plastics ,pulps ,slurries ,elmulsions, ect..) which do not obey the assumption of Newtonian fluid that the stress tensor is directing proportional to the deformation tensor ,are found in various engineering applications .An important class of non-Newtonian fluids is in the viscoelastic Oldroyd-B fluid has been intensively studied , and it has been widely applied to flow problems of small relaxation and retardation times.

Rheological constitutive equations with fractional derivatives have been proved to be a valuable tool to handle viscoelastic properties . The fractional derivative models of the viscoelastic fluids are derived from the classical equations which are modified by replacing the time derivative of an integer order by precise non-integer order integrals or derivatives.

In recent years , the Oldroyd –B has been acquired a special status amongst the many fluids of the rate type ,as it includes as special case the classical Newtonian fluid and Maxwell fluid.As a result of their wide implications ,a lot of papers regarding these fluids have been published in the last time .

The Oldroyd –B fluid model [11], which takes into account elastic and memory effects exhibited by most polymeric and biological liquids, has been used quite widely [4] .Existence ,uniqueness and stability results for some shearing motions of such a fluid have been obtained in[13] . the exact solution for the flow of an Oldroyd –B fluid was established by Waters and Kings [14], Rajagopal and Bhatnager [12],Fetecau [3], and Fetecau [2], other analytical results were given by Georgious [8] for small one- dimensional perturbations and for the limiting case of zero Reynold number unsteady(unidirectional and rotating)transient flows of an Oldroyd –B fluid in an annular areobtained by Tong[7] .thegeneral case of helical flow of Oldroyd –B fluid due to combine action of rotating cylinders(with constant angular velocities) and a constant axial pressure gradient has consider byWood [16].The velocity fields and the associated tangential stresses corresponding to helical flows of Oldroyd –B fluids using forms of series in term of Bessel functions are given by Fetecau et al [1].

Recently , the velocity field ,shear stress and vortex sheet of a generalized second –order fluid with fractional , fractional derivative using to the constitutive relationship models of Maxwell viscoelastic fluid and second order ,and some unsteady flows of a viscoelastic fluid and of second order fluids between two parallel platesare examined by Mingyo and wenchang [17].Unidirectionalflows of a viscoelastics fluid with the fractional Maxwell model helical flows of a generalized Oldroyd-B fluid with fractional calculus between two infinite coaxial circular cylinders are investigated by Dengke [6].

In this paper ,we study the effect of MHD on the helical flows of a generalized Oldroyd-B fluid with fractional calculus between two infinite coaxial circular cylinders. The velocity fields and the resulting shear stresses are determined by means of Laplace and finite Hankel transform and are presented under integral and series forms in the Mittag –leffler function.

II. Basic governing equations of Helcial flow between concentric cylinders1

We consider here an unsteady helical flow between two infinite coaxial cylinders located at $r = R_1$ and $r = R_2$ $(R_1 < R_2)$ in the cylindrical coordinates (r, θ, z) , the helcial velocity is given by

$$\boldsymbol{V} = r\, v(r,t) e_\theta + w(r,t) e_z \qquad (1)$$

is called helical , because its streamlines are helical ande_θ and e_z are the unit vectors in the θ and $z-$ directions , respectively . Since the velocity field is independent of θ and z and the constraint of incompressibility is automatically satisfied .

The constitutive equation of generalized Oldroyd-B (G Oldroyd-B) fluid has the form [1]

$$\boldsymbol{S} + \lambda_1^\alpha \frac{\delta^\alpha \boldsymbol{S}}{\delta t^\alpha} = \mu\left(1 + \lambda_2^\beta \frac{\delta^\beta}{\delta t^\beta}\right)\boldsymbol{A}_1 \qquad (2)$$

Where $\boldsymbol{S}$ is the extra stress tensor , μ is the dynamic viscosity, $\lambda_1$and λ_2 are material time constants referred to, the characteristic relaxation and characteristic retardation times , respectively . it is assumed that $\lambda_1 \geq \lambda_2 \geq 0$.$\boldsymbol{A}_1 = L + L^T$ is the first Rivlin –Erickscn tensor with L the velocity gradient , α and $\beta$are fractional calculus parameters such that $0 \leq \alpha \leq \beta \leq 1$ and the fractional operator $\frac{\delta^\alpha \boldsymbol{S}}{\delta t^\alpha}$ on any tensor $\boldsymbol{S}$ is defined by

$$\frac{\delta^\alpha \boldsymbol{S}}{\delta t^\alpha} = D_t^\alpha \boldsymbol{S} + \boldsymbol{V}.\nabla\boldsymbol{S} - L\boldsymbol{S} - \boldsymbol{S}\, L^T \qquad (3)$$

The operator D_t^α based on Caputo's fractional differential of order α is defined as

$$D_t^\alpha[y(t)] = \frac{1}{\Gamma(n-a)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau\,,$$

$$n - 1 < \alpha \leq n \qquad (4)$$

where $\Gamma(.)$denotes the Gamma function. This model can be reduced to the ordinary Maxwell fluid model when $\alpha = 1$,to a generalized theMaxwell fluid model when $\beta = 1$ and to an ordinary Oldroyd-B model $\beta = \alpha = 1$.

$$v(r,0) = w(r,0) = 0 \text{ and } \boldsymbol{S}(r,0) = 0 \qquad (5)$$

Here we assume that the generalized Oldroyd-B fluid is incompressible then

$$\nabla.\boldsymbol{V} = 0 \qquad (6)$$

Substituting Eq.(1) into Eq.(2) and Eq.(3) and taking into account (5), we find that $\boldsymbol{S}_{rr} = 0$, $\boldsymbol{\tau}(r,t) = \boldsymbol{S}_{r\theta}(r,t)$ is the shear stress and

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{r\theta} = \mu\left(1 + \lambda_2^\beta D_t^\beta\right)\left(r\frac{\partial \mathrm{v}}{\partial r}\right) \qquad (7)$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{rz} = \mu\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial w}{\partial r}\right) \qquad (8)$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{\theta z} = \lambda_1^\alpha\left[\left(r\frac{\partial \mathrm{v}}{\partial r}\right)\boldsymbol{S}_{rz} + \frac{\partial w}{\partial r}\boldsymbol{S}_{r\theta}\right] - 2\mu\,\lambda_2^\beta \frac{\partial w}{\partial r}\left(r\frac{\partial \mathrm{v}}{\partial r}\right) \qquad (9)$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{\theta\theta} = 2\lambda_1^\alpha\left(r\frac{\partial \mathrm{v}}{\partial r}\right)\boldsymbol{S}_{r\theta} - 2\mu\,\lambda_2^\beta\left(r\frac{\partial \mathrm{v}}{\partial r}\right)^2 \qquad (10)$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{zz} = 2\lambda_1^\alpha \frac{\partial w}{\partial r}\boldsymbol{S}_{rz} - 2\mu\,\lambda_2^\beta\left(\frac{\partial w}{\partial r}\right)^2 \qquad (11)$$

III. Momentum and continuity equation

We will write the formula of the momentum equation which governing the magnetohydrodynamic as fallows :

$$\rho\frac{d\boldsymbol{V}}{dt} = -\nabla p + \nabla.\boldsymbol{S} + \boldsymbol{J}\times\boldsymbol{B} \qquad (12)$$

Whereρ is the density of the fluid , $\boldsymbol{J}$ is the current density and $\boldsymbol{B} = [0, \beta_0, 0]$is the total magnetic field .

In the absence of body forces and a pressure gradient, the equation of motion reduce to the relevant equations

$$\rho\frac{\partial w}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\boldsymbol{S}_{rz} - \sigma\beta_0^2 w \qquad (13)$$

$$\rho r\frac{\partial \mathrm{v}}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\boldsymbol{S}_{r\theta} - \sigma\beta_0^2 r\mathrm{v} \qquad (14)$$

$$\frac{\partial p}{\partial r} + \frac{\boldsymbol{S}_{\theta\theta}}{r} = \rho r\mathrm{v}^2 \qquad (15)$$

Where σ is the electric conductivity .

Eliminating $\boldsymbol{S}_{rz}$ and $\boldsymbol{S}_{r\theta}$ among Eqs(7),(8),(13) and(14) ,we attain to the governing equations are

$$\rho\frac{\partial w}{\partial t} = \mu\frac{\left(1 + \lambda_2^\beta D_t^\beta\right)}{(1 + \lambda_1^\alpha D_t^\alpha)}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \sigma\beta_0^2 w \qquad (16)$$

$$\rho\, r\frac{\partial \mathrm{v}}{\partial t} = \mu\frac{\left(1 + \lambda_2^\beta D_t^\beta\right)}{(1 + \lambda_1^\alpha D_t^\alpha)}\left(r\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r}r\frac{\partial \mathrm{v}}{\partial t}\right) - \sigma\beta_0^2 r\mathrm{v} \qquad (17)$$

Multiply above two equations by $(1 + \lambda_1^\alpha D_t^\alpha)$,we get

$$\rho(1 + \lambda_1^\alpha D_t^\alpha)\frac{\partial w}{\partial t} = \mu\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \sigma\beta_0^2(1 + \lambda_1^\alpha D_t^\alpha)w \qquad (18)$$

$$\rho\, r(1 + \lambda_1^\alpha D_t^\alpha)\frac{\partial \mathrm{v}}{\partial t} = \mu\left(1 + \lambda_2^\beta D_t^\beta\right)\left(r\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r}r\frac{\partial \mathrm{v}}{\partial t}\right) - \sigma\beta_0^2 r(1 + \lambda_1^\alpha D_t^\alpha) \qquad (19)$$

Divide Eq(18) by ρ and divide Eq(19) by $\rho\, r$,we get

$$(1 + \lambda_1^\alpha D_t^\alpha)\frac{\partial w}{\partial t} = \upsilon\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \frac{\sigma\beta_0^2}{\rho}(1 + \lambda_1^\alpha D_t^\alpha)w \qquad (20)$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\frac{\partial \mathrm{v}}{\partial t} = \upsilon\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{v}}{\partial t}\right) - \frac{\sigma\beta_0^2}{\rho}(1 + \lambda_1^\alpha D_t^\alpha)\mathrm{v} \qquad (21)$$

Where $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid.

The boundary conditions are expressed by

$$w(R_1,t) = U_1\,,\quad w(R_2,t) = U_2\,,\ t > 0 \qquad (22)$$

And

$$\mathrm{v}(R_1,t) = \Omega_1\,,\quad \mathrm{v}(R_2,t) = \Omega_2\,,\ t > 0 \qquad (23)$$

The initial conditions are expressed by

$$w(r,0) = \mathrm{v}(r,0) = 0 \qquad (24)$$

$$\partial_t w(r,0) = \partial_t \mathrm{v}(r,0) = 0 \qquad (25)$$

3.1.Calculation of the velocity field

Making the change of unknown function

$$\mathrm{v}(r,t) = \frac{u(r,t)}{r} \qquad (26)$$

substitute the value of velocity $\mathrm{v}(r,t)$ in Eq. (21) with initial and boundary conditions ,we get

$$(1 + \lambda_1^\alpha D_t^\alpha)\frac{1}{r}\frac{\partial \mathrm{u}}{\partial t} = \frac{\upsilon}{r}\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 \mathrm{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{u}}{\partial t} - \frac{u}{r^2}\right) - \frac{\sigma\beta_0^2}{\rho r}(1 + \lambda_1^\alpha D_t^\alpha)\mathrm{u} \qquad (27)$$

Multiply above equation by r ,we get

$$(1+\lambda_1^\alpha D_t^\alpha)\frac{\partial u}{\partial t}=\nu(1+\lambda_2^\beta D_t^\beta)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial t}-\frac{u}{r^2}\right)-\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha D_t^\alpha)u \qquad (28)$$

And

$$u(R_1,t)=R_1\Omega_1,\quad u(R_2,t)=R_2\Omega_2,\quad t>0 \qquad (29)$$

$$u(r,0)=\partial_t u(r,0)=0 \qquad (30)$$

To obtain the exact analytical solution of the above problems Eq.(20) and Eq.(28) , and using initial conditions(24),(25) and (30) ,we first apply Laplace transform of fractional derivatives , with respect to t ,we get

$$s(1+\lambda_1^\alpha s^\alpha)\overline{w}=\nu(1+\lambda_2^\beta s^\beta)\left(\frac{\partial^2\overline{w}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{w}}{\partial r}\right)-\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\overline{w} \qquad (31)$$

$$\overline{w}(R_1,s)=\frac{U_1}{s},\quad \overline{w}(R_2,s)=\frac{U_2}{s} \qquad (32)$$

$$s(1+\lambda_1^\alpha s^\alpha)\overline{u}=\nu(1+\lambda_2^\beta s^\beta)\left(\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}}{\partial r}-\frac{\overline{u}}{r^2}\right)-\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\overline{u} \qquad (33)$$

$$\overline{u}(R_1,s)=\frac{R_1\Omega_1}{s},\quad \overline{u}(R_2,s)=\frac{R_2\Omega_2}{s} \qquad (34)$$

The solutions in Laplace space for above problems are given

$$\overline{w}(r,s)=\frac{U_1\psi_{0,0}(r,R_2,x(s))-U_2\psi_{0,0}(r,R_1,x(s))}{s\psi_{0,0}(R_1,R_2,x(s))} \qquad (35)$$

$$\overline{u}(r,s)=\frac{R_1\Omega_1\psi_{1,1}(r,R_2,x(s))-R_2\Omega_2\psi_{1,1}(r,R_1,x(s))}{s\psi_{1,1}(R_1,R_2,x(s))} \qquad (36)$$

$$\overline{v}(r,s)=\frac{R_1\Omega_1\psi_{1,1}(r,R_2,x(s))-R_2\Omega_2\psi_{1,1}(r,R_1,x(s))}{rs\psi_{1,1}(R_1,R_2,x(s))} \qquad (37)$$

Where $x(s)=\sqrt{\frac{\left(s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right)}{\nu(1+\lambda_2^\beta s^\beta)}}$, and

$\psi_{m,n}(a,b,y)=K_m(ay)I_n(by)+(-1)^{m+n+1}I_m(ay)K_n(by)$,K_m and I_n are the Bessel functions of the first and second kind , respectively.

The velocity field of helical flow between concentric cylinder is obtained by applying the Stehfest's method [9,10] of the numerical inversion of Laplace transform to Eqs,(35)and(36).

We use the finite Hankel transform with respect to r [5],defined as follows

$$\overline{\overline{w}}=\int_{R_1}^{R_2} r\,\overline{w}(r,s)\psi_1(s_{1n}r)dr \qquad (38)$$

$$\overline{\overline{u}}=\int_{R_1}^{R_2} r\,\overline{u}(r,s)\psi_2(s_{2n}r)dr \qquad (39)$$

And the inverse Hankel transform are

$$\overline{w}(r,s)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_{1n}^2J_0^2(s_{1n}R_1)\overline{\overline{w}}(r,s)\psi_1(s_{1n}r)}{J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2)} \qquad (40)$$

$$\overline{u}(r,s)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_{2n}^2J_1^2(s_{2n}R_1)\overline{\overline{u}}(r,s)\psi_2(s_{2n}r)}{J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2)} \qquad (41)$$

Where s_{1n} and s_{2n} are the positive roots of $\psi_1(s_{1n}R_1)=0$and $\psi_2(s_{2n}R_1)=0$,respectively.

$\psi_1(s_{1n}r)=Y_0(s_{1n}R_2)J_0(s_{1n}r)-J_0(s_{1n}R_2)Y_0(s_{1n}r)$,
$\psi_2(s_{2n}r)=Y_1(s_{2n}R_2)J_1(s_{2n}r)-J_1(s_{2n}R_2)Y_1(s_{2n}r)$

Y_i and J_i are the Bessel functions of the first and second kinds of order zero and one (i=0,1) , respectively.

Now applying finite Hankel transform to Eqs.(40) and (41) ,we get

$$\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{1n}^2\,\nu(1+\lambda_2^\beta s^\beta)\right]\overline{\overline{w}}=\frac{2}{\pi}\nu(1+\lambda_2^\beta s^\beta)\frac{U_2J_0(s_{1n}R_1)-U_1J_0(s_{1n}R_2)}{s\,J_0(s_{1n}R_1)} \qquad (42)$$

$$\overline{\overline{w}}=\frac{2\nu(1+\lambda_2^\beta s^\beta)[U_2J_0(s_{1n}R_1)-U_1J_0(s_{1n}R_2)]}{\pi s\,J_0(s_{1n}R_1)}\times\frac{1}{\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{1n}^2\,\nu(1+\lambda_2^\beta s^\beta)\right]} \qquad (43)$$

And

$$\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\,\nu(1+\lambda_2^\beta s^\beta)\right]\overline{\overline{u}}=\frac{2}{\pi}\nu(1+\lambda_2^\beta s^\beta)\frac{R_2\Omega_2J_1(s_{2n}R_1)-R_1\Omega_1J_1(s_{2n}R_2)}{s\,J_1(s_{2n}R_1)} \qquad (44)$$

$$\overline{\overline{u}}=\frac{2\nu(1+\lambda_2^\beta s^\beta)[R_2\Omega_2J_1(s_{2n}R_1)-R_1\Omega_1J_1(s_{2n}R_2)]}{\pi s\,J_1(s_{2n}R_1)}\times\frac{1}{\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\,\nu(1+\lambda_2^\beta s^\beta)\right]} \qquad (45)$$

Substitute Eq.(43) and (45) into (40) and (41) ,we obtain

$$\overline{w}(r,s)=\pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_1(s_{1n}r)[U_2J_0(s_{1n}R_1)-U_1J_0(s_{1n}R_2)]}{J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2)}\times\bar{A}_1(s_{1n},s) \qquad (46)$$

where

$$\bar{A}_1(s_{1n},s)=\frac{s_{1n}^2\,\nu(1+\lambda_2^\beta s^\beta)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{1n}^2\,\nu(1+\lambda_2^\beta s^\beta)\right]} \qquad (47)$$

And

$$\overline{u}(r,s)=\pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\psi_2(s_{2n}r)[R_2\Omega_2J_1(s_{2n}R_1)-R_1\Omega_1J_1(s_{2n}R_2)]}{J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2)}\times\bar{A}_1(s_{2n}, \qquad (48)$$

where

$$\bar{A}_1(s_{2n},s)=\frac{s_{2n}^2\,\nu(1+\lambda_2^\beta s^\beta)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\,\nu(1+\lambda_2^\beta s^\beta)\right]} \qquad (49)$$

Now ,rewrite Eqs.(47) and (48) in series form as

$$\bar{A}_1(s_{1n},s)=\frac{1}{s}-\left(s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right)\times\sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{m!}{b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{1n}^2\nu)^{c+d}\left(\lambda_2^\beta\right)^d s^\delta}{(\lambda_1^\alpha)^{m-b+1}}\times\frac{1}{\left(\frac{\sigma\beta_0^2}{\rho}\lambda_1^{-\alpha}s^{-1}+s^\alpha+\lambda_1^{-\alpha}\right)^{m+1}} \qquad (50)$$

Where $\delta=-m-2+\alpha*b+\beta*d.$

And

$$\bar{A}_1(s_{1n},s) = \frac{1}{s} - \left(s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right) \times \sum_{m=0}^{\infty}(-1)^m \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{m!}{b!\,c!\,d!} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{b} (s_{2n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d s^\delta}{(\lambda_1^\alpha)^{m-b+1}} \times \frac{1}{\left(\frac{\sigma\beta_0^2}{\rho}\lambda_1^{-\alpha}s^{-1} + s^\alpha + \lambda_1^{-\alpha}\right)^{m+1}} \quad (51)$$

Now , applying inverse Laplace transform to Eqs.(50) and (51) , we obtain

$$A_1(s_{1n},t) = 1 - \left(s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right) \times \sum_{m=0}^{\infty}(-1)^m \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{m!}{b!\,c!\,d!} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{b} (s_{1n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} \times L^{-1}\left(\frac{s^\delta}{\left(\frac{\sigma\beta_0^2}{\rho}\lambda_1^{-\alpha}s^{-1} + s^\alpha + \lambda_1^{-\alpha}\right)^{m+1}}\right) \quad (52)$$

And

$$A_1(s_{2n},t) = 1 - \left(s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right) \times \sum_{m=0}^{\infty}(-1)^m \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{m!}{b!\,c!\,d!} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{b} (s_{2n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} \times L^{-1}\left(\frac{s^\delta}{\left(\frac{\sigma\beta_0^2}{\rho}\lambda_1^{-\alpha}s^{-1} + s^\alpha + \lambda_1^{-\alpha}\right)^{m+1}}\right) \quad (53)$$

We will use the following property of Mittag-leffler function [5]

$$L^{-1}\left\{\frac{n!\,s^{\mu-\upsilon}}{(s^\mu \mp c)}\right\} = t^{\mu n+\upsilon-1}E_{\mu,\upsilon}^{(n)}(\pm ct^\mu)(Re(s) > |c|^{1/\mu}) \quad (54)$$

Then Eqs.(52) and (53) , are become

$$A_1(s_{1n},t) = 1 - \left((t+\lambda_1^\alpha t^{\alpha+1}) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha t^\alpha)\right) \times \sum_{m=0}^{\infty}(-1)^m \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{m!}{b!\,c!\,d!} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{b} (s_{1n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} \times t^{\alpha m+(\alpha-\delta)-1}E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) \quad (55)$$

and

$$A_1(s_{2n},t) = 1 - \left((t+\lambda_1^\alpha t^{\alpha+1}) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha t^\alpha)\right) \times \sum_{m=0}^{\infty}(-1)^m \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{m!}{b!\,c!\,d!} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{b} (s_{2n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} \times t^{\alpha m+(\alpha-\delta)-1}E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) \quad (56)$$

Substitute Eqs.(55) and (56) into (46) and (48) ,we get

$$w(r,t) = \pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_1(s_{1n}r)[U_2J_0(s_{1n}R_1) - U_1J_0(s_{1n}R_2)]}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} \times \left(1 - G_1(s_{1n},t)\right) \quad (57)$$

Where

$$G_1(s_{1n},t) = \sum_{m=0}^{\infty}(-1)^m \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{b}}{b!\,c!\,d!} \frac{(s_{1n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} * t^{\alpha m+(\alpha-\delta)-1}\left\{tE_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + \lambda_1^\alpha t^{\alpha+1}E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + \frac{\sigma\beta_0^2}{\rho}E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + \frac{\sigma\beta_0^2}{\rho}\lambda_1^\alpha t^\alpha E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right\} \quad (58)$$

And

$$u(r,t) = \pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\psi_2(s_{2n}r)[R_2\Omega_2J_1(s_{2n}R_1) - R_1\Omega_1J_1(s_{2n}R_2)]}{J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)} \times \left(1 - G_1(s_{2n},t)\right) \quad (59)$$

Where

$$G_1(s_{2n},t) = \sum_{m=0}^{\infty}(-1)^m \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{b}}{b!\,c!\,d!} \frac{(s_{2n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} * t^{\alpha m+(\alpha-\delta)-1}\left[tE_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + \lambda_1^\alpha t^{\alpha+1}E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + \frac{\sigma\beta_0^2}{\rho}E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + \frac{\sigma\beta_0^2}{\rho}\lambda_1^\alpha t^\alpha E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right] \quad (60)$$

Finally ,the inverse finite Hankel transform on $W(r)$ and $U(r)$

$$H(W(r)) = \int_{R_1}^{R_2} r\left[U_2 + \left(\frac{In\left(\frac{r}{R_2}\right)}{In\left(\frac{R_2}{R_1}\right)}\right)*(U_2-U_1)\right]\psi_1(s_{1n}r)dr = \frac{2[U_2J_0(s_{1n}R_1) - U_1J_0(s_{1n}R_2)]}{\pi s_{1n}^2 J_0(s_{1n}R_1)}$$

Where $W(r) = \left[U_2 + \left(\frac{In\left(\frac{r}{R_2}\right)}{In\left(\frac{R_2}{R_1}\right)}\right)*(U_2-U_1)\right]$,

We get

$$w(r,t) = W(r) - \pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_1(s_{1n}r)}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} \times [U_2J_0(s_{1n}R_1) - U_1J_0(s_{1n}R_2)]G_1(s_{1n},t) \quad (61)$$

$$H\big(U(r)\big) = \int_{R_1}^{R_2} r\left[r\Omega_2 + \left(\frac{R_1^2(r^2-R_2^2)}{r(R_2^2-R_1^2)}\right) * (\Omega_2-\Omega_1)\right]\psi_2(s_{2n}r)dr = \frac{2[R_2\Omega_2 J_1(s_{2n}R_1) - R_1\Omega_1 J_1(s_{2n}R_2)]}{\pi s_{2n}^2 J_1(s_{2n}R_1)}$$

Where $U(r) = \left[r\Omega_2 + \left(\frac{R_1^2(r^2-R_2^2)}{r(R_2^2-R_1^2)}\right) * (\Omega_2-\Omega_1)\right]$, we get

$$u(r,t) = U(r) - \pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\psi_2(s_{2n}r)}{J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)} \times [R_2\Omega_2 J_1(s_{2n}R_1) - R_1\Omega_1 J_1(s_{2n}R_2)]G_1(s_{2n},t) \quad (62)$$

3.2.Calculation of the tangential tensions and normal tensions

Applying the Laplace transform to Eqs.(7) and (8) , and using the initial conditions
$w(r,0) = \mathrm{u}(r,0) = 0$
$\partial_t w(r,0) = \partial_t \mathrm{u}(r,0) = 0$
we find that

$$\bar{\boldsymbol{S}}_{rz} = \frac{\mu(1+\lambda_2^\beta s^\beta)}{(1+\lambda_1^\alpha s^\alpha)}\left(\frac{\partial\bar{w}}{\partial r}\right) \quad (63)$$

$$\bar{\boldsymbol{S}}_{r\theta} = \frac{\mu(1+\lambda_2^\beta s^\beta)}{(1+\lambda_1^\alpha s^\alpha)}\left(r\frac{\partial\bar{\mathrm{v}}}{\partial r}\right) \quad (64)$$

Differentiating the Eqs. (35) and (36) with respect to r, we obtain

$$\frac{\partial\bar{w}}{\partial r} = \frac{U_2\psi_{1,0}\big(r,R_1,x(s)\big) - U_1\psi_{1,0}\big(r,R_2,x(s)\big)}{s\psi_{0,0}\big(R_1,R_2,x(s)\big)} \quad (65)$$

$$r\frac{\partial\bar{\mathrm{v}}}{\partial r} = \frac{\partial\bar{\mathrm{u}}}{\partial r} - \frac{\bar{\mathrm{u}}}{r} = \frac{R_2\Omega_2\psi_{1,1}\big(r,R_1,x(s)\big) - R_1\Omega_1\psi_{1,1}\big(r,R_2,x(s)\big)}{rs\psi_{1,1}\big(R_1,R_2,x(s)\big)} +$$

$\Big\{\Big\{R_2\Omega_2\Big[-\psi_{0,1}\big(r,R_1,x(s)\big) + \frac{1}{r}\psi_{1,1}\big(r,R_1,x(s)\big)\Big] -$
R1Ω1−ψ0,1r,R2,xs+1*rψ1,1r,R2,xs/rsψ1,1R1,R2,xs*
(66)

Substitute Eqs.(65) and (66) into Eqs(63) and (64) ,respectively ,we get

$$\bar{\boldsymbol{S}}_{rz} = \frac{\mu(1+\lambda_2^\beta s^\beta)}{s(1+\lambda_1^\alpha s^\alpha)}\left(\frac{U_2\psi_{1,0}\big(r,R_1,x(s)\big) - U_1\psi_{1,0}\big(r,R_2,x(s)\big)}{\psi_{0,0}\big(R_1,R_2,x(s)\big)}\right) \quad (67)$$

$$\bar{\boldsymbol{S}}_{r\theta} = \frac{\mu(1+\lambda_2^\beta s^\beta)}{s(1+\lambda_1^\alpha s^\alpha)}\Bigg(\frac{R_2\Omega_2\psi_{1,1}\big(r,R_1,x(s)\big) - R_1\Omega_1\psi_{1,1}\big(r,R_2,x(s)\big)}{r\psi_{1,1}\big(R_1,R_2,x(s)\big)} + \Big\{\Big\{R_2\Omega_2\Big[-\psi_{0,1}\big(r,R_1,x(s)\big) + \frac{1}{r}\psi_{1,1}\big(r,R_1,x(s)\big)\Big] - R_1\Omega_1\Big[-\psi_{0,1}\big(r,R_2,x(s)\big) + \frac{1}{r}\psi_{1,1}\big(r,R_2,x(s)\big)\Big]\Big\} /\big\{r\psi_{1,1}\big(R_1,R_2,x(s)\big)\big\}\Big\}\Bigg) \quad (68)$$

$$\bar{\boldsymbol{S}}_{rz} = \rho\left(\frac{U_2\psi_{1,0}\big(r,R_1,x(s)\big) - U_1\psi_{1,0}\big(r,R_2,x(s)\big)}{x_1^2(s)\psi_{0,0}\big(R_1,R_2,x(s)\big)}\right) \quad (69)$$

$$\bar{\boldsymbol{S}}_{r\theta} = \rho\Bigg(\frac{R_2\Omega_2\psi_{1,1}\big(r,R_1,x(s)\big) - R_1\Omega_1\psi_{1,1}\big(r,R_2,x(s)\big)}{rx_1^2(s)\psi_{1,1}\big(R_1,R_2,x(s)\big)} + \Big\{\Big\{R_2\Omega_2\Big[-\psi_{0,1}\big(r,R_1,x(s)\big) + \frac{1}{r}\psi_{1,1}\big(r,R_1,x(s)\big)\Big] - R_1\Omega_1\Big[-\psi_{0,1}\big(r,R_2,x(s)\big) + \frac{1}{r}\psi_{1,1}\big(r,R_2,x(s)\big)\Big]\Big\} /\big\{rx_1^2(s)\psi_{1,1}\big(R_1,R_2,x(s)\big)\big\}\Big\}\Bigg) \quad (70)$$

Where $x_1^2(s) = \frac{s(1+\lambda_1^\alpha s^\alpha)}{\upsilon(1+\lambda_2^\beta s^\beta)}$.

Differentiating the Eqs. (42) and (55) with respect to r, we obtain

$$\frac{\partial\bar{w}}{\partial r} = \sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)s_{1n}\psi_3(s_{1n}r)[U_2J_0(s_{1n}R_1) - U_1J_0(s_{1n}R_2)]}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} \times \frac{s_{1n}^2\,\upsilon(1+\lambda_2^\beta s^\beta)}{s\left[s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha) + s_{1n}^2\,\upsilon(1+\lambda_2^\beta s^\beta)\right]} \quad (71)$$

Where $\psi_3(s_{2n}r) = Y_0(s_{2n}R_2)J_1(s_{2n}r) - J_0(s_{2n}R_2)Y_1(s_{2n}r)$

$$.\frac{\partial\bar{\mathrm{v}}}{\partial r} = \frac{\partial\bar{\mathrm{u}}}{\partial r} - \frac{\bar{\mathrm{u}}}{r} = \pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\left(s_{2n}\psi_3(s_{2n}r) - \frac{1}{r}\psi_2(s_{2n}r)\right)}{J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)}[R_2\Omega_2J_1(s_{2n}R_1) - R_1\Omega_1J_1(s_{2n}R_2)] \times \frac{s_{2n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s\left[s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha) + s_{2n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \quad (72)$$

Substitute Eqs(71) and (72) into Eqs.(63) and (64), respectively then we obtain

$$\bar{\boldsymbol{S}}_{rz} = \frac{\mu\left(1+\lambda_2^\beta s^\beta\right)}{(1+\lambda_1^\alpha s^\alpha)}\left(\pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)s_{1n}\psi_3(s_{1n}r)}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} \times \frac{[U_2J_0(s_{1n}R_1) - U_1J_0(s_{1n}R_2)]s_{1n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s\left[s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha) + s_{1n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]}\right) \quad (73)$$

$$\bar{\boldsymbol{S}}_{r\theta} = \frac{\mu(1+\lambda_2^\beta s^\beta)}{(1+\lambda_1^\alpha s^\alpha)}\left(\pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\left(s_{2n}\psi_3(s_{2n}r) - \frac{1}{r}\psi_2(s_{2n}r)\right)}{(J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2))} \times \frac{[R_2\Omega_2J_1(s_{2n}R_1) - R_1\Omega_1J_1(s_{2n}R_2)]s_{2n}^2\,\upsilon(1+\lambda_2^\beta s^\beta)}{s\left[s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha) + s_{2n}^2\,\upsilon(1+\lambda_2^\beta s^\beta)\right]}\right) \quad (74)$$

Applying the Laplace transform to Eqs(73) and (74),we get

$$\boldsymbol{S}_{rz} = \left(\rho\pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_3(s_{1n}r)[U_2J_0(s_{1n}R_1) - U_1J_0(s_{1n}R_2)]}{s_{1n}\left(J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)\right)} \times L^{-1}\left\{\frac{s_{1n}^4\,\upsilon^2\left(1+\lambda_2^\beta s^\beta\right)^2}{s\left[s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha) + s_{1n}^2\,\upsilon(1+\lambda_2^\beta s^\beta)\right]}\right\}\right) \times L^{-1}\left\{\frac{1}{(1+\lambda_1^\alpha s^\alpha)}\right\} \quad (75)$$

$$\boldsymbol{S}_{r\theta} = \left(\pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\big(rs_{2n}\psi_3(s_{2n}r) - \psi_2(s_{2n}r)\big)}{rs_{2n}^2\,(J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2))}[R_2\Omega_2J_1(s_{2n}R_1) - R_1\Omega_1J_1(s_{2n}R_2)] \times L^{-1}\left\{\frac{1}{(1+\lambda_1^\alpha s^\alpha)}\right\} \times L^{-1}\left\{\frac{s_{2n}^4\,\upsilon^2\left(1+\lambda_2^\beta s^\beta\right)^2}{s\left[s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha) + s_{2n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]}\right\}\right) \quad (76)$$

Then Eqs(75) and (76) are become

$$\boldsymbol{S}_{rz} = \rho\pi\left(\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_3(s_{1n}r)[U_2J_0(s_{1n}R_1)-U_1J_0(s_{1n}R_2)]}{s_{1n}(J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2))}\times G_2(s_{1n},t)\right) \quad (77)$$

where

$$G_2(s_{1n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{\substack{b,c,d\geq 0\\ b+c+d=m}}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}}{b!\,c!\,d!}\frac{(s_{1n}^2\,\upsilon)^{2+c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} * t^{\alpha m+(2\alpha-\delta)-2}\left[E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)+2\lambda_2^\beta t^\beta E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)+\lambda_2^{2\beta}t^{2\beta}E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right]E_{\alpha,\alpha}(-\lambda_1^\alpha t^\alpha)$$

$$\boldsymbol{S}_{r\theta} = \rho\pi\left(\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\left(rs_{2n}\psi_3(s_{2n}r)-\psi_2(s_{2n}r)\right)}{rs_{2n}^2\left(J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2)\right)}[R_2\Omega_2J_1(s_{2n}R_1)-R_1\Omega_1J_1(s_{2n}R_2)]\times G_2(s_{2n},t)\right) \quad (78)$$

where

$$G_2(s_{2n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{\substack{b,c,d\geq 0\\ b+c+d=m}}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}}{b!\,c!\,d!}\frac{(s_{2n}^2\,\upsilon)^{2+c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} * t^{\alpha m+(2\alpha-\delta)-2}\left[E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)+2\lambda_2^\beta t^\beta E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)+\lambda_2^{2\beta}t^{2\beta}E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right]E_{\alpha,\alpha}(-\lambda_1^\alpha t^\alpha)$$

Let

$$F_{1,1}(r,t)=\lambda_1^\alpha\left[\left(r\frac{\partial \mathrm{v}}{\partial r}\right)\boldsymbol{S}_{rz}+\frac{\partial w}{\partial t}\,\boldsymbol{S}_{r\theta}\right]-2\mu\lambda_2^\beta\frac{\partial w}{\partial t}\left(r\frac{\partial \mathrm{v}}{\partial r}\right)$$

$$F_{1,2}(r,t)=2\lambda_1^\alpha\left(r\frac{\partial \mathrm{v}}{\partial r}\right)\boldsymbol{S}_{r\theta}-2\mu\,\lambda_2^\beta\left(r\frac{\partial \mathrm{v}}{\partial r}\right)^2$$

$$F_{1,3}(r,t)=2\lambda_1^\alpha\frac{\partial w}{\partial r}\boldsymbol{S}_{rz}-2\mu\,\lambda_2^\beta\left(\frac{\partial w}{\partial r}\right)^2$$

We get

$$(1+\lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{\theta z}=F_{1,1}(r,t) \quad (79a)$$

$$(1+\lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{\theta\theta}=F_{1,2}(r,t) \quad (79b)$$

$$(1+\lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{zz}=F_{1,3}(r,t) \quad (79c)$$

Applying Laplace transform to$(79a)$, $(79b)$ and (79c) ,we get

$$\overline{\boldsymbol{S}}_{\theta z}=\frac{\bar{F}_{1,1}(r,t)}{(1+\lambda_1^\alpha s^\alpha)} \quad (80a)$$

$$\overline{\boldsymbol{S}}_{\theta\theta}=\frac{\bar{F}_{1,2}(r,t)}{(1+\lambda_1^\alpha s^\alpha)} \quad (80b)$$

$$\overline{\boldsymbol{S}}_{zz}=\frac{\bar{F}_{1,3}(r,t)}{(1+\lambda_1^\alpha s^\alpha)} \quad (80c)$$

The inverse of Laplace transform to$(80a)$, $(80b)$ and (80c) ,can be expressed by

$$\boldsymbol{S}_{\theta z}=\frac{1}{\lambda_1^\alpha}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1^\alpha(t-\tau)^\alpha)F_{1,1}(r,t)\ d\tau \quad (81a)$$

$$\boldsymbol{S}_{\theta\theta}=\frac{1}{\lambda_1^\alpha}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1^\alpha(t-\tau)^\alpha)F_{1,2}(r,t)\ d\tau \quad (81b)$$

$$\boldsymbol{S}_{zz}=\frac{1}{\lambda_1^\alpha}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_1^\alpha(t-\tau)^\alpha)F_{1,3}(r,t)\ d\tau \quad (81c)$$

The definition of the weiessenberg number $W_i$is as follows

$$W_i=\frac{\boldsymbol{S}_{zz}-\boldsymbol{S}_{\theta\theta}}{\boldsymbol{S}_{zz}} \quad (82)$$

The shear stresses (the frictional force) i.e.,the drag exerted per unit length of the inner or outer cylinders or of the cylinder of radius $r=R_2$, in the second case ,can be calculated from Eqs.(73)and(74) , we get

$$\bar{\tau}_1=\overline{\boldsymbol{S}}_{rz}|_{r=R_2}=\frac{\mu\left(1+\lambda_2^\beta s^\beta\right)}{(1+\lambda_1^\alpha s^\alpha)}\left(\pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)s_{1n}\psi_3(s_{1n}R_2)}{J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2)}\right)\times[U_2\,J_0(s_{1n}R_1)-U_1\,J_0(s_{1n}R_2)]\times\frac{s_{2n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \quad (83)$$

$$\bar{\tau}_2=\overline{\boldsymbol{S}}_{r\theta}|_{r=R_2}=\frac{\mu\left(1+\lambda_2^\beta s^\beta\right)}{(1+\lambda_1^\alpha s^\alpha)}\times\left(\pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\left(s_{2n}\psi_3(s_{2n}r)-\frac{1}{r}\psi_2(s_{2n}r)\right)}{(J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2))}[R_2\Omega_2J_1(s_{2n}R_1)-R_1\Omega_1J_1(s_{2n}R_2)]\times\frac{s_{2n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]}\right) \quad (84)$$

Now ,we applying the inverse Laplace transform toEqs.(83)and(84) ,we obtain

$$\tau_1=\boldsymbol{S}_{rz}|_{r=R_2}=\rho\pi\left(\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_3(s_{1n}R_2)[U_2J_0(s_{1n}R_1)-U_1J_0(s_{1n}R_2)]}{s_{1n}(J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2))}\times G_2(s_{1n},t)\right) \quad (85)$$

$$\tau_2 = \boldsymbol{S}_{r\theta}|_{r=R_2} = \rho\pi\left(\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\left(rs_{2n}\psi_3(s_{2n}R_2)-\psi_2(s_{2n}R_2)\right)}{rs_{2n}^2\,(J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2))}[R_2\Omega_2J_1(s_{2n}R_1) - R_1\Omega_1J_1(s_{2n}R_2)]\times G_2(s_{2n},t)\right) \quad (86)$$

IV.Helical flow through circular cylinder I

Taking the limit of Eqs(38) and (39), when $R_1\to 0$ and $R_2\to R$, we find the Hankel transform

$$\bar{\bar{w}} = \int_0^R rJ_0(s_{3n}r)\,\bar{w}(r,s)dr \quad (87)$$

Where s_{3n} is positive root of $J_0(s_{3n}R)=0$, and

$$\bar{\bar{u}} = \int_0^R rJ_1(s_{4n}r)\,\bar{u}(r,s)dr \quad (88)$$

Where s_{4n} is positive root of $J_1(s_{4n}R)=0$, and the inverse Hankel transform are

$$\bar{w}(r,s) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\bar{\bar{w}}(s_{3n},s)J_0(s_{3n}r)}{J_1^2(s_{3n}R)} \quad (89)$$

and

$$\bar{u}(r,s) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\bar{\bar{u}}(s_{4n},s)J_1(s_{4n}r)}{J_2^2(s_{4n}R)} \quad (90)$$

Corresponding to the helical flow through an infinite circular cylinder . the boundary conditions must be changed by

$$|w(0,t)|<\infty \quad , \quad w(R,t)=U, \quad |u(0,t)|<\infty \;, u(R,t)=R\Omega \quad (91)$$

Now apply Laplace transform to the boundary conditions, with respect to t ,we get

$$|w(0,s)|<\infty, w(R,s)=\frac{U}{s}, \qquad |u(0,s)|<\infty \;,\; u(R,s)=\frac{R\Omega}{s} \quad (92)$$

Now applying finite Hankel transform to Eqs.(31) and (33) ,we get

$$\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{3n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]\bar{\bar{w}} = \upsilon\left(1+\lambda_2^\beta s^\beta\right)R\,s_{3n}\frac{U}{s}J_1(s_{3n}\,R) \quad (93)$$

then

$$\bar{\bar{w}} = \frac{\upsilon\left(1+\lambda_2^\beta s^\beta\right)R\,s_{3n}\,U\,J_1(s_{3n}\,R)}{s\;\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{3n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \quad (94)$$

and

$$\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{4n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]\bar{\bar{u}} = -\frac{R^2\Omega}{s}s_{4n}\upsilon\left(1+\lambda_2^\beta s^\beta\right)J_2(s_{4n}\,R) \quad (95)$$

Then

$$\bar{\bar{u}} = -\frac{R^2\Omega s_{4n}\upsilon\left(1+\lambda_2^\beta s^\beta\right)J_2(s_{4n}\,R)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{4n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \quad (96)$$

Substitute Eqs.(94) and(96) into Eqs(89) and (90)respectively and using the identity

$J_n'(s_{1n}\,r)=J_{n-1}(s_{1n}r)=-J_{n+1}(s_{1n}r)$

,we find that

$$\bar{w}(r,s)=\frac{2U}{R}\sum_{n=1}^{\infty}\frac{J_0(s_{3n}r)}{s_{3n}J_1(s_{3n}\,R)}\bar{A}_2(s_{3n},s) \quad (97)$$

where

$$\bar{A}_1(s_{3n},s) = \frac{s_{3n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{3n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \quad (98)$$

and

$$\bar{u}(r,s)=2\Omega\sum_{n=1}^{\infty}\frac{J_1(s_{4n}r)}{s_{4n}J_0(s_{4n}\;R)}\bar{A}_2(s_{4n},s) \quad (99)$$

where

$$\bar{A}_1(s_{4n},s) = \frac{s_{4n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{4n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \quad (100)$$

Now ,rewrite Eqs.(98)and (100) in series form as

$$\bar{A}_1(s_{3n},s) = \frac{1}{s}-\left(s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right)\times\sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{m!}{b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{3n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d s^\delta}{(\lambda_1^\alpha)^{m-b+1}}\times\frac{1}{\left(\frac{\sigma\beta_0^2}{\rho}\lambda_1^{-\alpha}s^{-1}+s^\alpha+\lambda_1^{-\alpha}\right)^{m+1}} \quad (101)$$

Where $\delta=-m-2+\alpha*b+\beta*d$, and

$$\bar{A}_1(s_{4n},s) = \frac{1}{s}-\left(s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right)\times\sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{m!}{b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{4n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d s^\delta}{(\lambda_1^\alpha)^{m-b+1}}\times\frac{1}{\left(\frac{\sigma\beta_0^2}{\rho}\lambda_1^{-\alpha}s^{-1}+s^\alpha+\lambda_1^{-\alpha}\right)^{m+1}} \quad (102)$$

Now , applying inverse Laplace transform to Eqs.(101)and(102) , we obtain

$$A_1(s_{3n},t) = 1-\left(s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right)\times\sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{m!}{b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{3n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}}\times L^{-1}\left(\frac{s^\delta}{\left(\frac{\sigma\beta_0^2}{\rho}\lambda_1^{-\alpha}s^{-1}+s^\alpha+\lambda_1^{-\alpha}\right)^{m+1}}\right) \quad (103)$$

$$A_1(s_{4n},t) = 1-\left(s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)\right)\times\sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{m!}{b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{4n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}}\times L^{-1}\left(\frac{s^\delta}{\left(\frac{\sigma\beta_0^2}{\rho}\lambda_1^{-\alpha}s^{-1}+s^\alpha+\lambda_1^{-\alpha}\right)^{m+1}}\right) \quad (104)$$

Now ,we will use the property(54) of Mittag-leffler function [5] ,Then Eqs.(103)and (104) ,are become

$$A_1(s_{3n},t) = 1-\left((t+\lambda_1^\alpha t^{\alpha+1})+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha t^\alpha)\right)\times\sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{m!}{b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{3n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}}$$

$$\times t^{\alpha m+(\alpha-\delta)-1}E^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) \quad (105)$$

and

$$A_1(s_{4n},t) = 1-\left((t+\lambda_1^\alpha t^{\alpha+1})+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha t^\alpha)\right)$$
$$\times\sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{m!}{b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{4n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}}$$
$$\times t^{\alpha m+(\alpha-\delta)-1}E^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) \quad (106)$$

Substitute Eqs.(105) and (106) into (97) and (99) ,we get

$$w(r,t)=\frac{2U}{R}\sum_{n=1}^{\infty}\frac{J_0(s_{3n}r)}{s_{3n}J_1(s_{3n}\,R)}\,(1-\;G_1(s_{3n},t)) \quad (107)$$

where

$$G_1(s_{3n},t) = \sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}}{b!\,c!\,d!}\frac{(s_{3n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}}$$
$$* t^{\alpha m+(\alpha-\delta)-1}\left[tE^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right.$$
$$+\lambda_1^\alpha t^{\alpha+1}E^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)$$
$$+\frac{\sigma\beta_0^2}{\rho}E^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)$$
$$\left.+\frac{\sigma\beta_0^2}{\rho}\lambda_1^\alpha t^\alpha E^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right] \quad (108)$$

and

$$u(r,t)=2\Omega\sum_{n=1}^{\infty}\frac{J_1(s_{4n}r)}{s_{4n}J_0(s_{4n}\,R)}\,(1-\;G_1(s_{4n},t)\,) \quad (109)$$

where

$$G_1(s_{4n},t) = \sum_{m=0}^{\infty}(-1)^m\sum_{b+c+d=m}^{b,c,d\geq 0}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}}{b!\,c!\,d!}\frac{(s_{4n}^2\,\upsilon)^{c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}}$$
$$* t^{\alpha m+(\alpha-\delta)-1}\left[tE^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right.$$
$$+\lambda_1^\alpha t^{\alpha+1}E^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)$$
$$+\frac{\sigma\beta_0^2}{\rho}E^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)$$
$$\left.+\frac{\sigma\beta_0^2}{\rho}\lambda_1^\alpha t^\alpha E^{(m)}_{\alpha,(\alpha-\delta)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right] \quad (110)$$

Finally ,the inverse finite Hankel transform on $W(r)$and $U(r)$, we get

$$H(W(r))=\int_0^R rJ_0(s_{3n}r)\,Udr=\frac{UR}{s_{3n}}J_1(s_{3n}R)$$

where $W(r)=U$, we get

$$w(r,t)=U-\frac{2U}{R}\sum_{n=1}^{\infty}\frac{J_0(s_{3n}r)}{s_{3n}J_1(s_{3n}\,R)}G_1(s_{3n},t) \quad (111)$$

and

$$H(U(r))=\int_0^R rJ_1(s_{4n}r)\,2\Omega dr\;=\frac{2\Omega}{s_{4n}}J_2(s_{4n}R)=\frac{-2\Omega}{s_{4n}}J_0(s_{4n}R)$$

Where $U(r)=-2\Omega$, we get

$$u(r,t)=-2\Omega+2\Omega\sum_{n=1}^{\infty}\frac{J_1(s_{4n}r)}{s_{4n}J_0(s_{4n}\,R)}G_1(s_{4n},t) \quad (112)$$

4.1.Calculation of the tangential tensions and normal tensions

Differentiating the Eqs. (97) and (99) with respect to r, and substitute into Eqs. (63) and (64) respectively, we obtain

$$\overline{\boldsymbol{S}}_{rz} =\frac{\mu\left(1+\lambda_2^\beta s^\beta\right)}{(1+\lambda_1^\alpha s^\alpha)}\left(-\frac{2U}{R}\sum_{n=1}^{\infty}\frac{J_1(s_{3n}r)}{s_{3n}^2\,J_1(s_{3n}\,R)}\times\frac{s_{3n}^4\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{3n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]}\right) \quad (113)$$

$$\overline{\boldsymbol{S}}_{r\theta} =\frac{\mu\left(1+\lambda_2^\beta s^\beta\right)}{(1+\lambda_1^\alpha s^\alpha)}\left(2\Omega\sum_{n=1}^{\infty}\frac{[(J_1(s_{4n}r)/r)-s_{4n}J_0(s_{4n}r)]}{rs_{4n}^3\,J_0(s_{4n}\,R)}\times\frac{-r\,s_{4n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{4n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]}\right) \quad (114)$$

Applying the Laplace transform to Eqs(113) and (114),we get

$$\boldsymbol{S}_{rz} =\left(-\frac{2\rho U}{R}\sum_{n=1}^{\infty}\frac{J_1(s_{3n}r)}{s_{3n}^2\,J_1(s_{3n}\,R)}\times L^{-1}\left\{\frac{s_{3n}^4\,\upsilon^2\left(1+\lambda_2^\beta s^\beta\right)^2}{s\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{3n}^2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]\times(1+\lambda_1^\alpha s^\alpha)}\right\}\right) \quad (115)$$

$$\boldsymbol{S}_{r\theta} = \left(2\rho\Omega \sum_{n=1}^{\infty} \frac{[(J_1(s_{4n}r)/r) - s_{4n}J_0(s_{4n}r)]}{r s_{4n}^3 J_0(s_{4n}R)} \times L^{-1}\left\{\frac{-r\, s_{4n}^2\, \upsilon^2 \left(1+\lambda_2^\beta s^\beta\right)^2}{s\left[s(1+\lambda_1^\alpha s^\alpha) + \frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha) + s_{4n}^2\, \upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]}\right\}\right) \times L^{-1}\left(\frac{1}{(1+\lambda_1^\alpha s^\alpha)}\right) \qquad (116)$$

Then Eqs(115) and (116) are become

$$\boldsymbol{S}_{rz} = -\frac{2\rho U}{R}\left(\sum_{n=1}^{\infty} \frac{J_1(s_{3n}r)}{s_{3n}^2 J_1(s_{3n}R)} \times G_2(s_{3n},t)\right) \qquad (117)$$

where

$$G_2(s_{3n},t) = \sum_{m=0}^{\infty}(-1)^m \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}}{b!\,c!\,d!} \frac{(s_{3n}^2\,\upsilon)^{2+c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} * t^{\alpha m+(2\alpha-\delta)-2}\left[E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + 2\lambda_2^\beta t^\beta E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + \lambda_2^{2\beta} t^{2\beta} E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right] E_{\alpha,\alpha}(-\lambda_1^\alpha t^\alpha)$$

and

$$\boldsymbol{S}_{r\theta} = \left(2\rho\Omega \sum_{n=1}^{\infty} \frac{\left[\left(\frac{J_1(s_{4n}r)}{r}\right) - s_{4n}J_0(s_{4n}r)\right]}{r s_{4n}^3 J_0(s_{4n}R)} \times G_2(s_{4n},t)\right) \qquad (118)$$

where

$$G_2(s_{4n},t) = \sum_{m=0}^{\infty}(-1)^{m+1} \sum_{b+c+d=m}^{b,c,d\geq 0} \frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}} r}{b!\,c!\,d!} \frac{(s_{4n}\upsilon)^{2+c+d}\left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} * t^{\alpha m+(2\alpha-\delta)-2}\left[E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + 2\lambda_2^\beta t^\beta E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right) + \lambda_2^{2\beta} t^{2\beta} E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^\alpha}\left(1+\frac{\frac{\sigma\beta_0^2}{\rho}}{t}\right)t^\alpha\right)\right] E_{\alpha,\alpha}(-\lambda_1^\alpha t^\alpha)$$

Let

$$F_{2,1}(r,t) = \lambda_1^\alpha\left[\left(r\frac{\partial \mathrm{v}}{\partial r}\right)\boldsymbol{S}_{rz} + \frac{\partial w}{\partial t}\boldsymbol{S}_{r\theta}\right] - 2\mu\lambda_2^\beta \frac{\partial w}{\partial t}\left(r\frac{\partial \mathrm{v}}{\partial r}\right)$$

$$F_{2,2}(r,t) = 2\lambda_1^\alpha\left(r\frac{\partial \mathrm{v}}{\partial r}\right)\boldsymbol{S}_{r\theta} - 2\mu\,\lambda_2^\beta\left(r\frac{\partial \mathrm{v}}{\partial r}\right)^2$$

$$F_{2,3}(r,t) = 2\lambda_1^\alpha \frac{\partial w}{\partial r}\boldsymbol{S}_{rz} - 2\mu\,\lambda_2^\beta\left(\frac{\partial w}{\partial r}\right)^2$$

We get

$$(1+\lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{\theta z} = F_{2,1}(r,t) \qquad (119a)$$
$$(1+\lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{\theta\theta} = F_{2,2}(r,t) \qquad (119b)$$
$$(1+\lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{zz} = F_{2,3}(r,t) \qquad (119c)$$

Applying Laplace transform t$o(119a)$, $(119b)$ and$(119c)$,we get

$$\bar{\boldsymbol{S}}_{\theta z} = \frac{F_{2,1}(r,t)}{(1+\lambda_1^\alpha s^\alpha)} \qquad (120a)$$
$$\bar{\boldsymbol{S}}_{\theta\theta} = \frac{F_{2,2}(r,t)}{(1+\lambda_1^\alpha s^\alpha)} \qquad (120b)$$
$$\bar{\boldsymbol{S}}_{zz} = \frac{F_{2,2}(r,t)}{(1+\lambda_1^\alpha s^\alpha)} \qquad (120c)$$

The inverse of Laplace transform t$o(120a)$, $(120b)$ and (120c) ,can be expressed by

$$\boldsymbol{S}_{\theta z} = \frac{1}{\lambda_1^\alpha}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_1^\alpha(t-\tau)^\alpha)F_{2,1}(r,t)\,d\tau \qquad (121a)$$

$$\boldsymbol{S}_{\theta\theta} = \frac{1}{\lambda_1^\alpha}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_1^\alpha(t-\tau)^\alpha)F_{2,2}(r,t)\,d\tau \qquad (121b)$$

$$\boldsymbol{S}_{zz} = \frac{1}{\lambda_1^\alpha}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_1^\alpha(t-\tau)^\alpha)F_{2,3}(r,t)\,d\tau \qquad (121c)$$

The definition of the weiessenberg number $W_i$is as follows

$$W_i = \frac{\boldsymbol{S}_{zz} - \boldsymbol{S}_{\theta\theta}}{\boldsymbol{S}_{zz}} \qquad (122)$$

The shear stresses (the frictional force) i.e.,the drag exerted per unit length of the inner or outer cylinders or of the cylinder of radius $r = R_2$, in the second case ,can be calculated from Eqs.(117)and(118) , we get

$$\tau_3 = \boldsymbol{S}_{rz}|_{r=R_2} = -\frac{2\rho U}{R}\left(\sum_{n=1}^{\infty}\frac{J_1(s_{3n}R_2)}{s_{3n}^2 J_1(s_{3n}R)} \times G_2(s_{3n},t)\right) \qquad (123)$$

$$\tau_4 = \boldsymbol{S}_{r\theta}|_{r=R_2} = 2\rho\Omega\left(\sum_{n=1}^{\infty}\frac{\left[\left(\frac{J_1(s_{4n}R_2)}{r}\right) - s_{4n}J_0(s_{4n}R_2)\right]}{R_2 s_{4n}^3 J_0(s_{4n}R)} \times G_2(s_{4n},t)\right) \qquad (124)$$

V. Limiting case

1- Making the limit of Eqs(61) and (63) when $\alpha \neq 0$, $\beta \neq 0$ and $\left(\frac{\sigma\beta_0^2}{\rho}\right) = M = 0$,we can attain the similar solution velocity distribution for unsteady helical flows of a generalized Oldroyd –B fluid , as obtained in [6] , thus velocity fields reduces to

$$w(r,t) = W(r) - \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\psi_1(s_{1n}r)[U_2 J_0(s_{1n}R_1) - U_1 J_0(s_{1n}R_2)]}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} \times G_1(s_{1n},t) \qquad (125)$$

where

$$G_1(s_{1n},t) = \sum_{m=0}^{\infty} (-1)^m \sum_{c+d=m}^{c,d\geq 0} \frac{(s_{1n}^2\,\upsilon)^{c+d}}{c!\,d!} \frac{(\lambda_2^\beta)^d}{(\lambda_1^\alpha)^{m+1}} * t^{\alpha m+(\alpha-\delta)-1} \left[t E_{\alpha,(\alpha-\delta)}^{(m)}(-\lambda_1^{-\alpha} t^\alpha) + \lambda_1^\alpha t^{\alpha+1} E_{\alpha,(\alpha-\delta)}^{(m)}(-\lambda_1^{-\alpha} t^\alpha) \right]$$

Or

$$G_1(s_{1n},t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} \left(\frac{s_{1n}^2\,\upsilon}{\lambda_1^\alpha}\right)^m \times \sum_{k=0}^{m} \binom{m}{k} (\lambda_2^\beta)^k\, t^{(1+\alpha)m-\beta k} E_{\alpha,(\alpha+m+1-\beta k)}^{(m-1)}(-\lambda_1^{-\alpha} t^\alpha)$$

And

$$u(r,t) = U(r) - \pi \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\psi_2(s_{2n}r)[R_2\Omega_2 J_1(s_{2n}R_1) - R_1\Omega_1 J_1(s_{2n}R_2)]}{J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)} \times G_1(s_{2n},t) \qquad (126)$$

Where

$$G_1(s_{2n},t) = \sum_{m=0}^{\infty} (-1)^m \sum_{c+d=m}^{c,d\geq 0} \frac{(s_{2n}^2\,\upsilon)^{c+d}}{c!\,d!} \frac{(\lambda_2^\beta)^d}{(\lambda_1^\alpha)^{m+1}} * t^{\alpha m+(\alpha-\delta)-1} \left[t E_{\alpha,(\alpha-\delta)}^{(m)}(-\lambda_1^{-\alpha} t^\alpha) + \lambda_1^\alpha t^{\alpha+1} E_{\alpha,(\alpha-\delta)}^{(m)}(-\lambda_1^{-\alpha} t^\alpha) \right]$$

Or

$$G_1(s_{1n},t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} \left(\frac{s_{2n}^2\,\upsilon}{\lambda_1^\alpha}\right)^m \sum_{k=0}^{m} \binom{m}{k} (\lambda_2^\beta)^k\, t^{(1+\alpha)m-\beta k} E_{\alpha,(\alpha+m+1-\beta k)}^{(m-1)}(-\lambda_1^{-\alpha} t^\alpha)$$

2- Making the limit of Eqs(77) and (78) when $\alpha \neq 0$, $\beta \neq 0$ and $\left(\frac{\sigma\beta_0^2}{\rho}\right) = M = 0$,we can attain the similar solution shear stress for unsteady helical flows of a generalized Oldroyd –B fluid , as obtained in [6] , thus shear stress reduces to

$$\boldsymbol{S}_{rz} = \rho\pi \left(\sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\psi_3(s_{1n}r)}{s_{1n}(J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2))} [U_2 J_0(s_{1n}R_1) - U_1 J_0(s_{1n}R_2)] \times G_2(s_{1n},t) \right) \qquad (127)$$

where

$$G_2(s_{1n},t) = \sum_{m=0}^{\infty} (-1)^m \sum_{c+d=m}^{c,d\geq 0} \frac{(s_{1n}^2\,\upsilon)^{2+c+d}}{c!\,d!} \frac{(\lambda_2^\beta)^d}{(\lambda_1^\alpha)^{m+1}} * t^{\alpha m+(2\alpha-\delta)-2} * \left[E_{\alpha,(\alpha-\delta)}^{(m)}(-\lambda_1^{-\alpha} t^\alpha) + 2\lambda_2^\beta t^\beta E_{\alpha,(\alpha-\delta)}^{(m)}(-\lambda_1^{-\alpha} t^\alpha) + \lambda_2^{2\beta} t^{2\beta} E_{\alpha,(\alpha-\delta)}^{(m)}(-\lambda_1^{-\alpha} t^\alpha) \right] E_{\alpha,\alpha}(-\lambda_1^\alpha t^\alpha)$$

Or

$$G_2(s_{1n},t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \left(\frac{s_{1n}^2\,\upsilon}{\lambda_1^\alpha}\right)^{m+2} \sum_{k=0}^{m} \binom{m+2}{k} (\lambda_2^\beta)^k \times t^{(1+\alpha)(m+2)-\beta k-1} E_{\alpha,(\alpha+m+2-\beta k)}^{(m+1)}(-\lambda_1^{-\alpha} t^\alpha)$$

$$\boldsymbol{S}_{r\theta} = \rho\pi \left(\sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\left(r s_{2n}\psi_3(s_{2n}r) - \psi_2(s_{2n}r)\right)}{r s_{2n}^2\,(J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2))} \times [R_2\Omega_2 J_1(s_{2n}R_1) - R_1\Omega_1 J_1(s_{2n}R_2)]\; G_2(s_{2n},t) \right) \qquad (128)$$

VI.Numerical result and conclusion

In this paper we established the effect of MHD flow of the unsteadyhelical flows of an Oldroyd-B fluid in concentric cylinders and circular cylinder. The exact solutionof the unsteadyhelical flows of an Oldroyd-B fluid in an annular for the velocity field u , w and the associated shear stresses τ_1 , τ_2 are obtained by using Hankel transform and Laplace transform for fractional calculus . Moreover , some figures are plotted to show the behavior of various parameters involved in the expression of velocities w ,u (Eqs(61 and 62)) , shear stresses τ_1 , τ_2(Eqs(85 and 86)) ,respectively .

All theresult in this section are plotting graph by using MATHEMATICA package .

Fig(1) is depicted to show the change of the velocity w(r,t) with the non-integer fractional parameter α . the velocity is decreasing with increase of α. Fig(2) is prepared to show the effect of the variations of fractional parameter β .the velocity field w is increasing with increasefractional parameter β .Fig(3) represents the variation of velocity w for different values of relaxation λ_1 . The velocity is decreasing with increase λ_1.Fig(4 and 5) are prepared to show the effect of the variation of retardation λ_2 and kinematic viscosity v on the velocity field .The velocity field w is increasing with increase the parameters λ_2 and v .Fig (6) is established value of magnetic parameter M .the velocity field is decreases with increasing of M .Fig(7) illustrate the influence of time t on the velocity field w. the velocity is increasing with increase the time t.

Fig(8) is provided the graphically illustrations for effects ofthe non-integer fractional parameter α on the velocity u(r,t) with the non-integer fractional parameter α . the velocity is decreasing with increase of α.Fig(9) is prepared to show the effect of the variations of fractional parameter β .the velocity field w is increasing with increasefractional parameter β .Fig(10) represents the variation of velocity w for different values of relaxation λ_1 . The velocity is decreasing with increase λ_1.Fig(11 and 12) are prepared to show the effect of the variation of retardation λ_2 and kinematic viscosity v on the velocity field .The velocity field w is increasing with increase the parameters λ_2 and v .Fig (13) is established value of magnetic parameter M .the velocity field is decreases with increasing of M .Fig(14) illustrate the influence of time t on the velocity field w. the velocity is increasing with increase the time t.

Figs(15-20) provide the graphically illustrations for the effects of (fractional parameters(α , β) , relaxation λ_1 , retardation λ_2 , kinematic viscosity v , magnetic parameter M and time t) on the shear stress τ_1 . The shear stress decreasing with increase the different parameters.

Figs(21-26) are established to show the behavior of the parameters (fractional parameters(α , β) , relaxation λ_1 , retardation λ_2 , kinematic viscosity v , magnetic parameter M and time t) on the shear stress τ_2 . The shear stress decreasing with increase the different parameters.

The velocity w

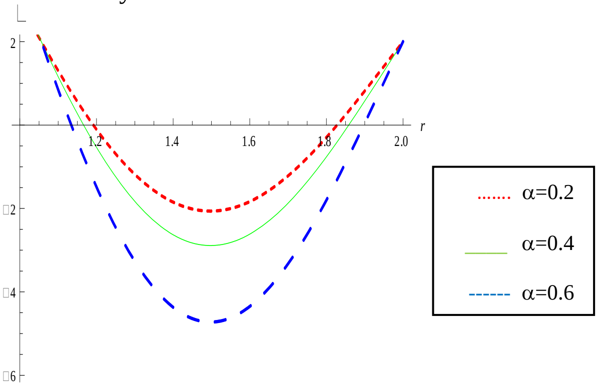

Fig1:- the velocity w for different value of fractional parameter α {λ1=15 , λ2=8 , *v*=0.165 , M=0, β=0.6 , K₁=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

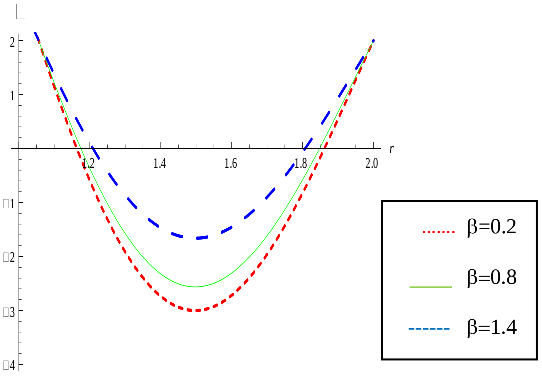

Fig2:- the velocity w for different value of fractional parameter β {λ1=15 , λ2=8 , *v*=0.165 , M=0.1,α=0.4 , K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

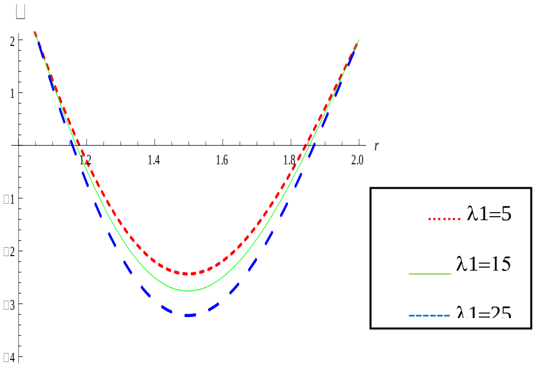

Fig3:- the velocity w for different value λ1 { λ2=8 , *v*=0.165 , M=0.1,α=0.4 , K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

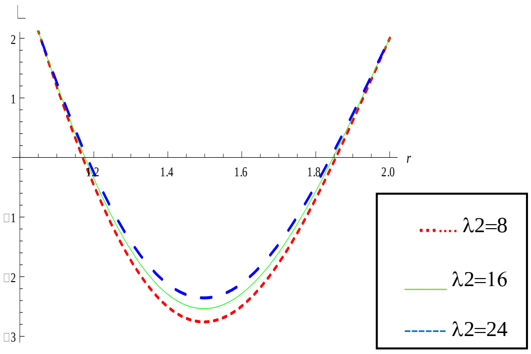

Fig4:- the velocity w for different value λ2 { λ1=15 , *v*=0.165 , M=0.1, β=0.6 , α=0.4 , K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

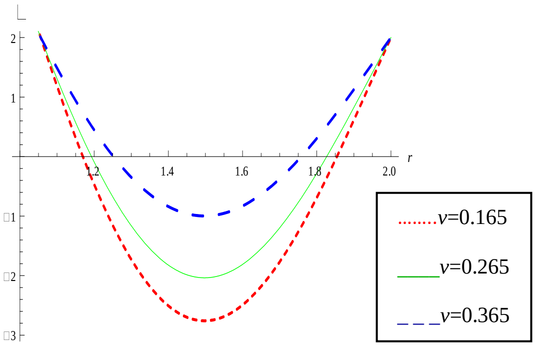

Fig5:- the velocity w for different value *v* {λ1=15 , λ2=8 , M=0.1 , β=0.6, α=0.4, K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

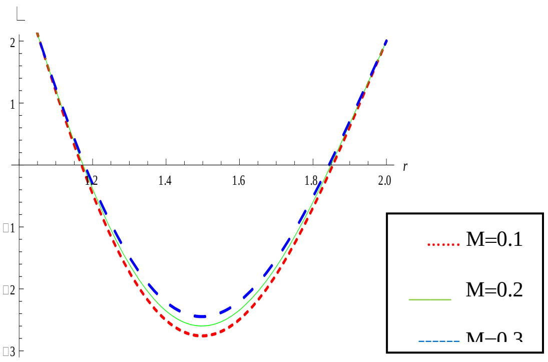

Fig6:- the velocity w for different value M { λ1=15 , λ2=8, *v*=0.165,α=0.4 , K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

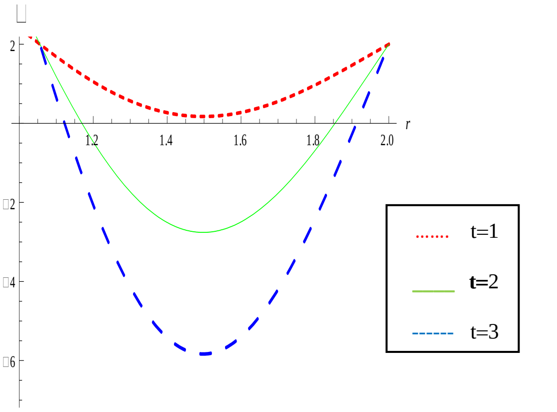

Fig7:- the velocity w for different value t { λ1=15 , λ2=8,M=0.1, *v*=0.165,α=0.4 , K_1=3.31114 , R1=1, R2=2, u1=2, u2=2}

The velocity of u

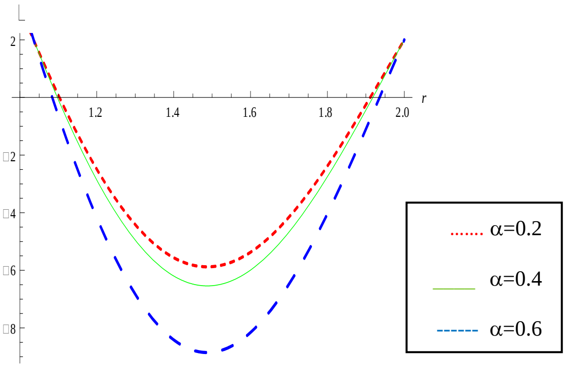

Fig8:- the velocity u for different value of fractional parameter α {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, β=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

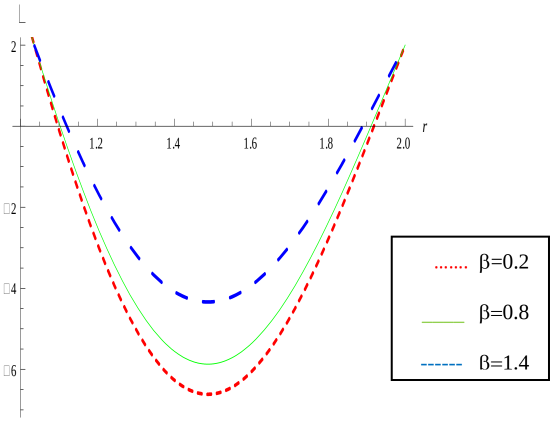

Fig9:- the velocity u for different value β {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, α=0.4, K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

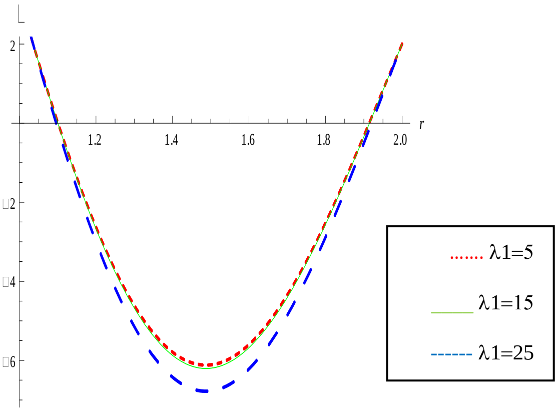

Fig10:- the velocity u for different value λ1 { λ2=8 , *v*=0.165 , M=0.1, α=0.4, β=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

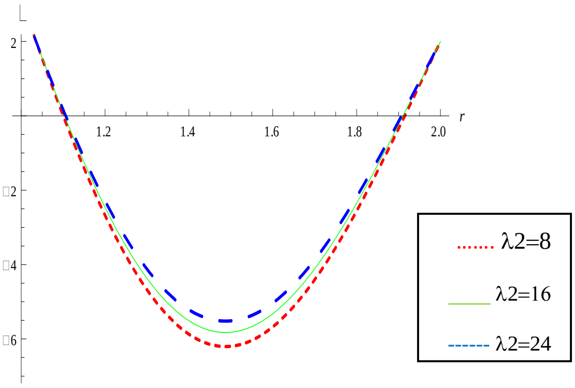

Fig11:- the velocity u for different value λ2 {λ1=15, *v*=0.165 , M=0.1, β=0.6 , α=0.4, K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

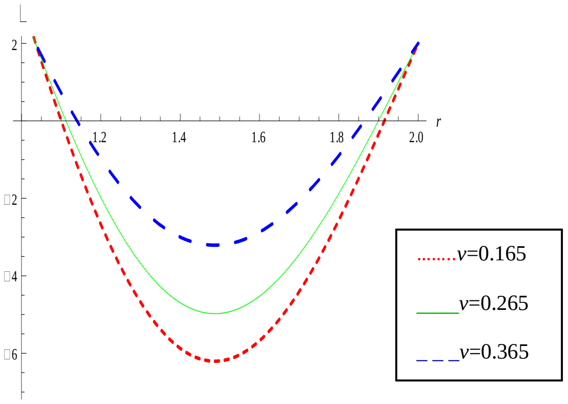

Fig12:- the velocity u for different value *v* {λ1=15 , λ2=8 , M=0.1 , β=0.6, α=0.4, K_1=3.31114 , t=1,R1=1, R2=2, u1=2, u2=2}

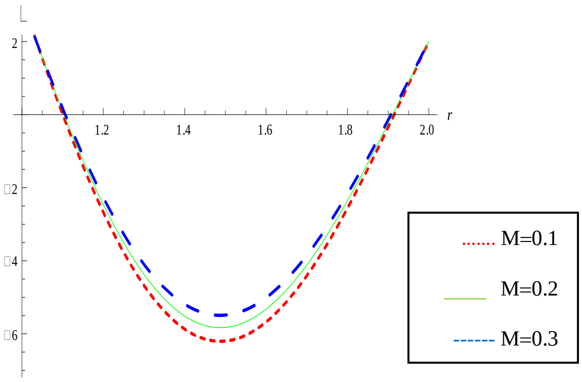

Fig13:- the velocity u for different value M **{λ1=15 , λ2=8 , *v*=0.165 , M=0.1, β=0.6 , α=0.4, K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}**

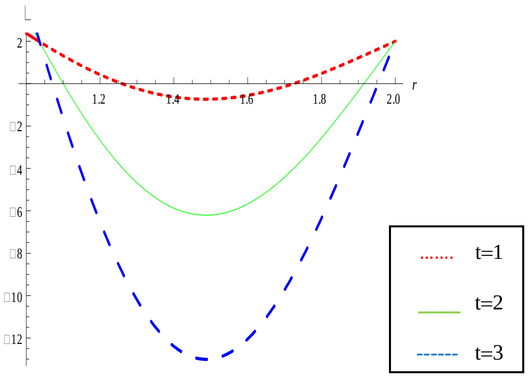

Fig14:- the velocity u for different value t {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, β=0.6 ,α=0.4, K_1=3.31114 , t=2,R1=1, R2=2, u1=2, u2=2}

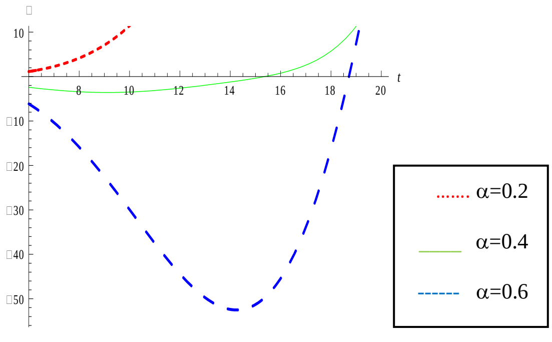

Fig .15. the shear stress for different value of fractional parameter α {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, β=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}

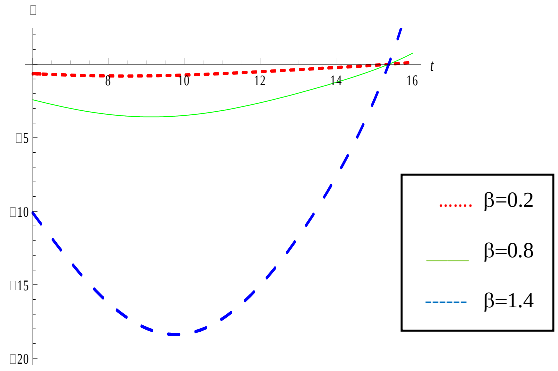

Fig .16. the shear stress for different value of fractional parameter β {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, α=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}

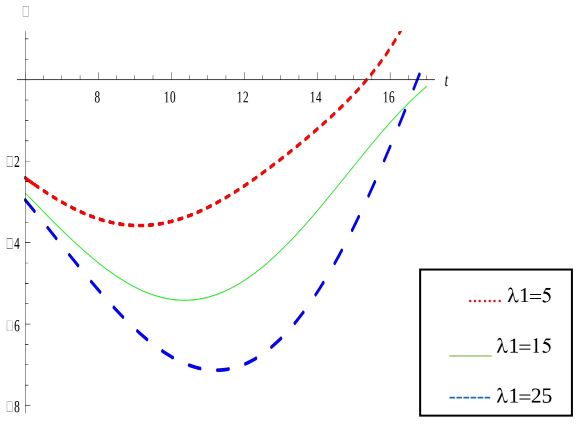

Fig .17. the shear stress for different value of λ1{λ2=8 , *v*=0.165 , M=0.1, β=0.6 , α=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}

The shree stress

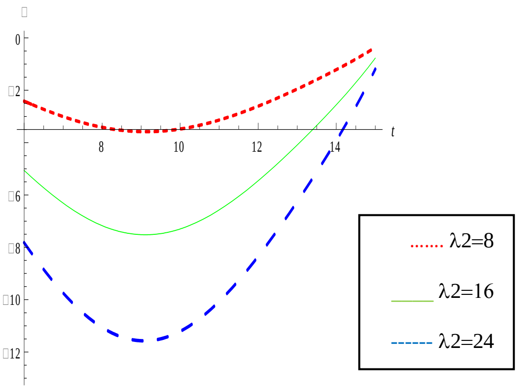

Fig .18. the shear stress for different value of λ2{λ1=15 , *v*=0.165 , M=0.1, β=0.6 , α=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}

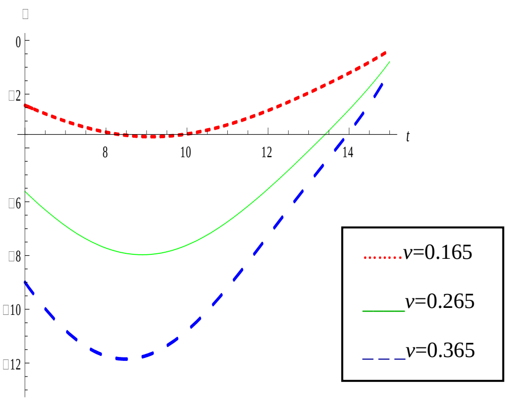

Fig19:- the the shear stress for different value *v* {λ1=15 , λ2=8 , M=0.1 , β=0.6, α=0.4, K_1=3.31114 , t=1,R1=1, R2=2, u1=2, u2=2,r=2,ρ=0.1}

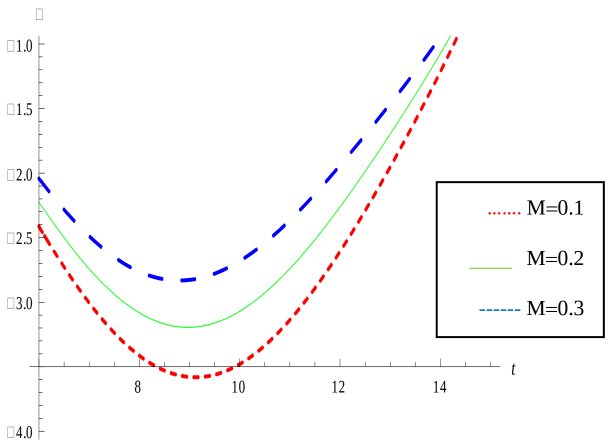

Fig .20. the shear stress for different value of M **{λ1=15 ,λ2=8, *v*=0.165 , M=0.1, β=0.6 , α=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}**

the sheer stress 2

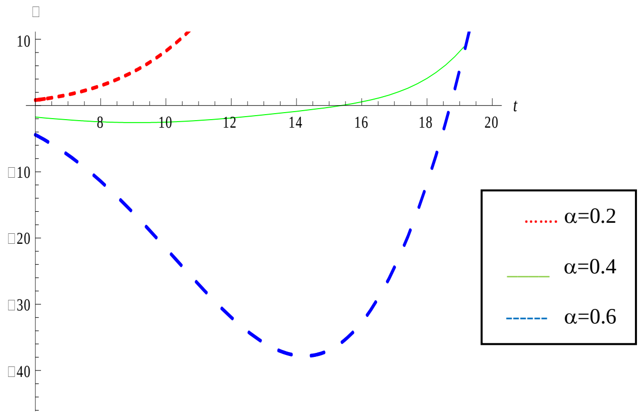

Fig .21 the shear stress for different value of fractional parameter α {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, β=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}

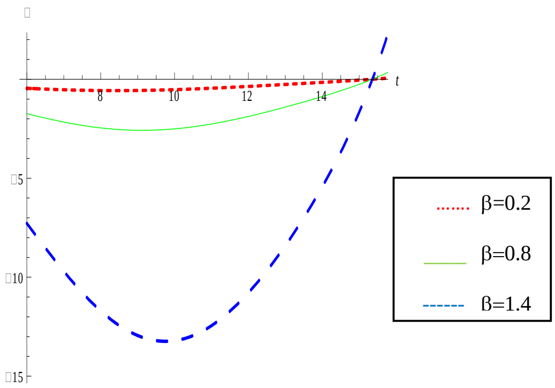

Fig .22. the shear stress for different value of fractional parameterβ {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, α=0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}

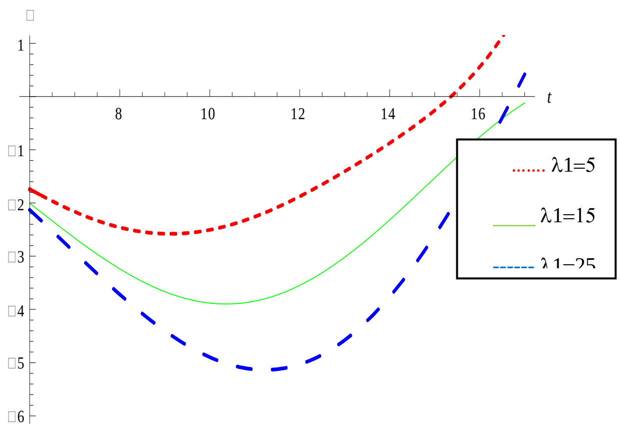

Fig .23. the shear stress for different value of λ1 { λ2=8 , *v*=0.165 , M=0.1 , β =0.6, α =0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}

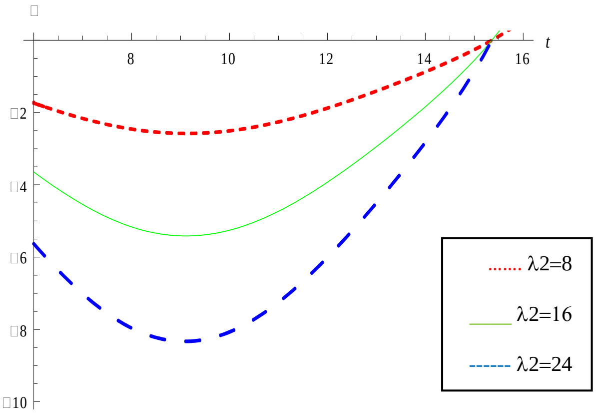

Fig .24. the shear stress for different value of λ2 { λ1=15 , *v*=0.165 , M=0.1 , β =0.6, α =0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}

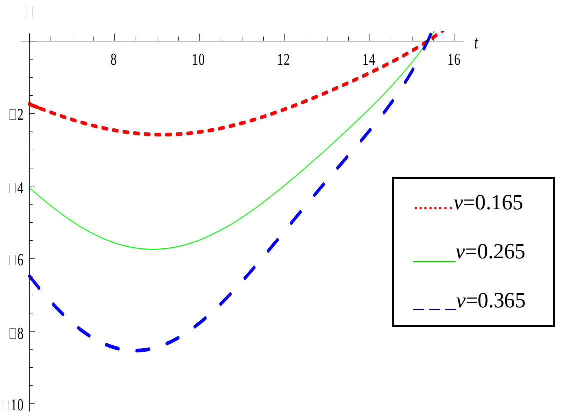

Fig25:- the the shear stress for different value *v* {λ1=15 , λ2=8 , M=0.1 , β=0.6, α=0.4, K_1=3.31114 , t=1,R1=1, R2=2, u1=2, u2=2,r=2,ρ=0.1}

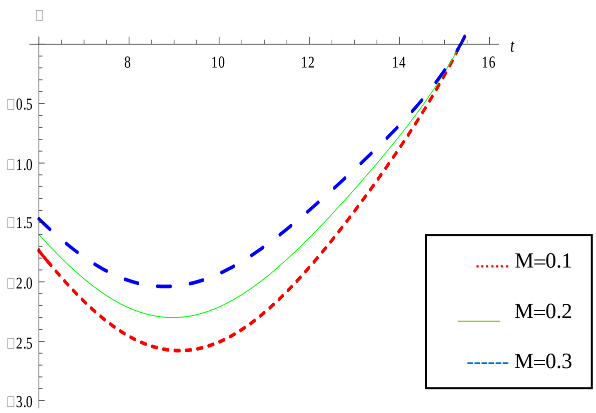

Fig .26. the shear stress for different value of M **{λ1=15 , λ2=8, *v*=0.165, β =0.6, α =0.6 , K_1=3.31114 , t=2,R1=1, R2=2, u1=1, u2=1,r=2, ρ=0.1}**

References

[1] C.Fetecau,C.Fetecau,D.Vieru ,On some helical flows of Oldroyd –B fluids, Acta Mech . 189 (2007) 53-63.
[2] C.Fetecau,C.Fetecau, The first problem of Stokes for an Oldroyd –B fluid, Int .J.Non –Linear Mech.38(2003) 1539-1544.
[3] C.Fetecau , The Rayleigh –Stokes problem for an edge in an Oldroyd –B fluid .C.R.Acad. Paris Ser.1335(2003) 979-984.
[4] C.Guilope ,J.c ,Saut ,Global existence and one-dimensional non –linear stability of shearing motion of viscoelastic fluids of Oldroyd type ,RAIRO Model ,Math .Anal ,Number 24)1990)369-401.
[5] -Debnath,l.and Bhatta,D.Integral transforms and Their Application Boca Rato ,London , Newyork , chapman &Hall.CRC;2007.
[6] Dengke Tong , Xianmin Zhang ,Xinhong Zhang ,Unsteady helical flows of a generalized Oldroyd-B fluid ,J.Non-Newtonian fluid Mech. 156 (2009) 75-83 .
[7] D.K.Tong,R.H.Wang ,Exact solution for the flow of non-Newtonian fluid with fractional derivative in an annular pipe , Sci .Chin .Sec G 48 (2005) 485-495
[8] G.C.Georgiou ,On the stability of the shear flow of a viscoelasic fluid with slip along the fixed wall ,Rheol .Acta 35 (1996) 39-47.
[9] H.Stechfest ,Algorithm 368 numerical inversion of Laplace transform ,Commum ,ACM 13(1) (1970) 47-49.
[10] H.Stechfest .Remark on Alrorithim 368numerical inversion of Laplace transform ,Commum ,ACM 13(10) (1970) 624-625.
[11] J.G. Oldroyd , on the formulation of rheological equation of state ,Proc,R.Soc . Lond .A 200(1950)523-541.
[12] K.R. Rajagopal ,P.K.Bahtngar .Exact solution for some simple flows of an Oldroyd-B fluid. Acta Mech 113(1995)233-239.
[13] M.A .Fontelos ,A.Friedman .Stationary non-Newtonian fluid flows in channel-like and pipe –like domains. Arch .Rational Mech .Anal 151(2001) 1-43.
[14] N.D.Waters M.J. King ,Unsteady flow of an elastico –viscous liquid ,Rheol .Acta 93(1970) 345-355.
[15] T.Wenchang ,P.Wenxian .X. Mingyu,Anote on unsteady flows of a viscoelastic fluids with the fractional Maxwell model between two parallel plates ,Int.J.Non –Linear Mech.38(5)(2003) 645-650.
[16] W.P.Wood , Transient viscoelastic helical flows in pipe of circular and annular cross –section ,J. Non -Newtonian fluid Mech.100(2001) 115-126.
[17] X.Mingyu,T.Wenchang ,Theoretical analysis of the velocity field , stress field and vortex sheet of generalized second order fluid with fractional anomalous diffusion ,Sci,Chin (Ser.A) 44(11)(2001) 1387-1399.